

Exact, stable, two-derivative interacting massless multi-graviton theories

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Abstract

We present a general model of interacting metric fields with the sum of massless non-interacting spin 2 fields as the linear limit. In the non-interacting limit the model is reduced to a sum of general relativity actions, with the usual Einstein-Hilbert kinetic term for each metric field.

Until now the accepted view was that such theories are inconsistent since it has been proven that ghost terms and/or discontinuity in the number of degrees of freedom at the zero interaction limit, are unavoidable. We do not refute these results, but instead prove by construction that they do not necessarily lead to inconsistencies. In particular, our suggested theories are:

1. Energetically stable, i.e. their Hamiltonian is bounded from below and the multi-Minkowski metric configuration is the unique ground state.
2. Continuous in the zero interaction limit so that general relativity solutions are restored smoothly.
3. Include no higher than two derivative Lagrangian terms.

In addition, the dominant energy condition is maintained with respect to all metric fields for all field configurations.

1 Introduction

Multi graviton theories are spin two massless fields that without interaction are described by a sum of Pauli-Fierz actions, which is a linearized form of a sum of Einstein-Hilbert actions, the free action for general relativity. The interaction involves, coupling of different metric fields and their derivatives, beside the usual metric coupling to standard matter. The interaction is restricted to maximum two derivative terms, i.e. addends with two first derivatives or one second derivative in order to keep the classical form of second order differential equations.

These theories were considered not consistent for a long time due to the impossibility to construct such interacting systems which avoid negative-propagating

excitations and at the same time preserving the full gauge symmetry of the free limit [1].

The appearance of ghost at some perturbation level may indicate energy instability. However, the absence of ghost degrees of freedom at any level of perturbation is not a necessary condition for stability.

In this paper we construct an energetically stable action that maintains the conditions of a generalized positive energy theorem [6], an expansion of the general relativity positive energy theorem[9] for several interacting metrics.

Breaking the symmetry of the free theory may lead to what appears to be a discontinuity of the solutions in the zero interaction limit, especially if the limit is taken in a finite perturbation order of the full theory. Our model is presented in an exact non-perturbative form, and the continuity at the zero interaction limit is guaranteed automatically. This can in principle be shown for all field equations by explicitly observing that no singularities occur, but it is sufficient to take a global point of view. In general, the Cauchy problem is well posed for multi-metric theories[4]. Then, propagation in time of the initial conditions is continuous in the interaction parameter. When this parameter goes to zero we get identity of some of the variables that are functionally independent for any finite value of the interaction parameter, so there are no conceptual problems concerning with elimination of degrees of freedom.

It should be noted that there are known models with interacting multi-graviton fields of which only one is massless, and therefore do not fit to our framework. These models include special interactions which yield a constraint that removes the ghost degree of freedom.[2]

Recently [5] we presented a model of energetically stable bimetric theories, for which the dominant energy condition is valid. In this paper we construct the action with two derivatives at the most and explain the continuity at the free limit, therefore provide a consistent interacting massless multi-metric theory. we also generalize these results to any number of cross-interacting metric fields and state the connection to immunity from causality violations.

2 The model

The general action is:

$$S = \int \mathcal{L} d^4x = \int \left(\sum_{a=1}^N \alpha^a \mathcal{L}_G^a + \mathcal{L}_I \right) d^4x \quad (1)$$

where α^a are non-negative constants, $\mathcal{L}_G^a = \frac{1}{16\pi G} R^a \sqrt{g^a}$ and $\mathcal{L}_I = \mathcal{L}_I[g_{\mu\nu}^1, \dots, g_{\mu\nu}^N, \psi, \nabla^1\psi, \dots, \nabla^N\psi]$. The scalar curvature R^a is derived from the metric $g_{\mu\nu}^a$ and $g^a \equiv -\det[g_{\mu\nu}^a]$. The fields ψ may be tensors of any rank, and ∇^a is the covariant derivative with respect to the metric $g_{\mu\nu}^a$. Any standard matter Lagrangian density is included in the interaction.

The numerical value of the constrained Hamiltonian can be calculated by any constrained Hamiltonian on the constraint surfaces, without solving the

constraints explicitly. A constrained Hamiltonian density is the zero-zero component of the Noether current we get from invariance with respect to coordinate translation:

$$\Theta_{\beta}^{\alpha} = -\mathcal{L}\delta_{\beta}^{\alpha} + \sum_a g^a_{lm,\alpha} \frac{\partial \mathcal{L}}{\partial g^a_{lm,\beta}} + \psi_{,\alpha} \frac{\partial \mathcal{L}}{\partial \psi_{,\beta}} \quad (2)$$

This is the canonical energy momentum pseudo-tensor for the system. It can be separated to the sum of gravity energy momentum pseudo-tensors for each metric, and an interaction term for which its space integration is equal to the space integration of the sum of all energy momentum tensors for each metric [6]

$$\int \left(-\mathcal{L}_I \delta_{\alpha}^0 + \left(\sum_a g^a_{lm,\alpha} \frac{\partial \mathcal{L}_I}{\partial g^a_{lm,0}} \right) + \psi_{,\alpha} \frac{\partial \mathcal{L}_I}{\partial \psi_{,0}} \right) d^3x = \int \sum_a \sqrt{-g^a} T^a_{\alpha}{}^0 d^3x \quad (3)$$

where

$$T^a_{\alpha}{}^{\beta} \equiv g^a_{\alpha\nu} \frac{2}{\sqrt{-g^a}} \frac{\delta \mathcal{L}_I}{\delta g^a_{\nu\beta}} \quad (4)$$

Using field equations and the Belinfante procedure for each metric [7] a , we get that up to space integration

$$\Theta^{\mu\nu} = \frac{1}{16\pi G} \frac{\partial}{\partial x^{\alpha}} \frac{\partial}{\partial x^{\beta}} \sum_a \alpha^a \sqrt{-g^a} (\eta^{\mu\nu} g^{a\alpha\beta} - \eta^{\alpha\nu} g^{a\mu\beta} + \eta^{\alpha\beta} g^{a\mu\nu} - \eta^{\mu\beta} g^{a\alpha\nu}) \quad (5)$$

That is, the total energy momentum tensor for the interacting multi-graviton system can be presented as a sum of divergence-free pseudo-tensors, each of which is a functional of only one metric. If the metric fields obeys the usual asymptotic conditions

$$g^a_{\mu\nu} \simeq \eta_{\mu\nu} + O(r^{-1}) \quad (6)$$

then the canonical energy is equal to a sum of ADM energy [8] expressions:

$$P^{\nu} = \sum_a P^{a\nu} \quad (7)$$

where

$$P^{a0} \equiv \frac{1}{16\pi G} \int \left(\frac{\partial g^a_{ij}}{\partial x^j} - \frac{\partial g^a_{jj}}{\partial x^i} \right) ds^i \quad (8)$$

and

$$P^{aj} \equiv \frac{1}{16\pi G} \int \left(\frac{\partial g^a_{kk}}{\partial x^0} \frac{\partial g^a_{k0}}{\partial x^k} \delta_{ij} + \frac{\partial g^a_{j0}}{\partial x^i} - \frac{\partial g^a_{ij}}{\partial x^0} \right) ds^i \quad (9)$$

The dominant energy condition (DEC) is defined by the requirement that $T_{\mu\nu}^a u^{a\mu} v^{a\nu} \geq 0$ for all time-like future-pointing vectors $u^{a\alpha}, v^{a\alpha}$, where time-likeness is defined with respect to the metric $g_{\mu\nu}^a$. If DEC is valid for all energy momentum tensors then the positive energy theorem [9, 10] can be applied for every index "a" and for the total energy momentum vector

$$P^{a0} \geq |P^{ai}| \Rightarrow P^0 \geq |P^i| \quad (10)$$

where equality is obtained when all the metric fields take the Minkowski matrix form and all (interaction) energy momentum tensors are zero.

We can guarantee DEC for every field configuration if the energy momentum tensors are conformal to their corresponding metrics

$$T_{\mu\nu}^a = F^a g_{\mu\nu}^a \quad (11)$$

and the scalars F^a are non-negative for all field configurations

$$F^a[g_{\mu\nu}^1, \dots, g_{\mu\nu}^N, \psi, \nabla^1 \psi, \dots, \nabla^N \psi] \geq 0 \quad (12)$$

We now focus on the case where the interaction has no explicit dependence on metric derivatives.

The energy momentum tensors must be integrated to the same interaction Lagrangian density, so

$$\frac{\delta^2 \mathcal{L}_I}{\delta g_{\mu\nu}^a \delta g_{\mu'\nu'}^b} = \frac{\partial \sqrt{g^a} T_{\mu\nu}^a}{\partial g^{b\mu'\nu'}} = \frac{\partial \sqrt{g^b} T_{\mu'\nu'}^b}{\partial g^{a\mu\nu}} \quad (13)$$

Because derivation of a metric determinant with respect to the metric is conformal to the metric, we choose the dependence of the interaction on the metric to be solely through the determinants.

One option is to choose energy momentum tensors which are symmetric in the metric fields

$$F^{a \in X} = \left(\sum_{b \in X} \sqrt{g^b} \right)^{-4k} \mathcal{D}[\psi, \psi, \rho] \quad (14)$$

$$F^{a \notin X} = 0$$

For some set $X \subset \{1, \dots, N\}$. The constant k is a positive integer and the functional \mathcal{D} is a non-negative scalar density with the weight of $4k$.

Integration of the energy momentum tensors gives:

$$\mathcal{L}_I^{(X,k)} = \int \frac{\sqrt{g^{a \in X}}}{2} T_{\mu\nu}^a dg^{a\mu\nu} = \frac{\mathcal{D}[\psi, \psi, \rho]}{1-4k} \left(\sum_{b \in X} \sqrt{g^b} \right)^{-4k+1} \quad (15)$$

This interaction can be generalized by a linear combination of the determinants with non-negative coefficients summation of all these possible interactions

$$\mathcal{L}_I^k = \mathcal{D}[\psi, \psi, \rho] \int_0^\infty f[\beta^1, \dots, \beta^N] \left(\sum_a \beta^a \sqrt{g^a} \right)^{-4k+1} \prod_a d\beta^a \quad (16)$$

The non-negative scalar density D may be an even power of the determinant of the matrix of some scalar field derivatives [5]. If we want to restrict the Lagrangian to at most two derivatives in each addend, we construct the scalar density with non-metric 2-rank tensor field:

$$\mathcal{D} \equiv \left(\det \left[\sum_b A_\alpha^b \otimes A_\beta^b \right] \right)^{2k} \quad (17)$$

where A_α^b is a vector field in the index α and $b = 1, \dots, N$ with $N \geq 4$ in order to make sure that the determinant of the sum of matrices is not identically zero.

We add a kinetic term for these vector fields that obey DEC:

$$\mathcal{L}_{IK}^a = - \sum_{b,c} \mu^{bc} F_{\alpha\beta}^b g^{c\mu\nu} g^{c\beta\nu} F_{\mu\nu}^b \sqrt{g^a} \quad (18)$$

where $F_{\alpha\beta}^b \equiv \partial_\alpha A_\beta^b - \partial_\beta A_\alpha^b$ and μ^{bc} are non-negative constants.

The energy momentum tensor that is derived from the kinetic term constitutes another source in the field equation for the metric $g_{\mu\nu}^a$, and it obeys DEC with respect to this specific metric for all field configurations. This can be proven for any anti-symmetric field $F_{\alpha\beta}$ and interaction in the form of (18), as in the case of electromagnetic field coupled to our standard metric.

The energy momentum tensors (11) have to obey asymptotic conditions $T_{\mu\nu}^a \simeq O(r^{-3-\varepsilon})$ in order to maintain the boundary conditions (6) for the metric fields. For any positive integer k one can assume the desired boundary conditions on the fields A_α and its first derivatives so they go to zero fast enough in order to maintain the required asymptotic conditions for the energy momentum tensors. These boundary conditions are consistent with the A_α field equations.

A general multi-graviton interaction for a stable system is then

$$\mathcal{L}_I = \sum_k \mathcal{L}_I^k + \sum_a \mathcal{L}_{IK}^a \quad (19)$$

where the addends are defined in eq. (16), (17), (18). According to the positive energy theorem for interacting multi-metric systems the Lagrangian in eq.(1) with the interaction (19) describe a stable multi-graviton theory with non-negative energy, where the lowest energy state is obtained when all the metric fields are Minkowskian $g_{\mu\nu}^a = \eta_{\mu\nu}$ and all energy momentum tensors are zero.

3 Discussion

The model presented in this paper, defined by the first and last equation, is a stable general interacting multi-graviton theory, with at most two derivative

terms is the Lagrangian.

In the construction of this non-negative energy model we did not use special interactions that remove ghost degrees of freedom, Thus our model avoids superluminal shock waves and possible causality problems which have been claimed for the ghost free single metric massive gravity and its bi-metric extension, based on that interaction.[3]

Furthermore, our model enjoys a strong immunity to causal problems in general, and not just from those subjected to the above special constraints. All energy momentum tensors obey DEC with respect to the metric for which they are the source in the field equations for that metric. According to Hawking [12] and Tipler [11], in general relativity, if the energy momentum tensor obeys the weak energy condition (WEC) then closed time-like curves (CTC) can be created in a compact region only if singularities are created also. DEC implies WEC, therefore CTC in any spacetime described by some metric field, where time-likeness is with respect to the same metric, can be created only if there are singularities in that spacetime. Indeed, singularity in these scenarios does not mean necessarily a point where the curvature is infinite. and the connection of the interaction to such a singularity is less obvious than to infinite curvature. These singularities can not be excluded simply by regularity characters of energy momentum tensors, as they do have in our model, and in fact there are models in general relativity with CTC in an asymptotically flat spacetime region with standard matter and finite curvature [13]. However, these singularity conditions definitely reduce the chance for CTC in our model, and even if one may construct a stable multi-metric theory with CTC, it does not constitute evidence for its illness any more than that the time-machine models mentioned above indicates illness in general relativity.

Our Lagrangian is composed from a general combination of interactions that include every subset of the graviton fields, thus the number of different metric fields on each vertex can take any value.

The model presented in this paper include the case of $N=1$, i.e. general relativity with modified self interaction, but not massive gravity. Massive terms are quadratic in the graviton fields. In order to maintain boundary conditions that are needed to define finite total energy the energy momentum tensors can not include such terms, assuming that standard matter may also included. Massive terms are either repulsive, i.e. contribute negative energy density therefore violate DEC, or with an opposite sign but do not suit appropriate boundary conditions. Traditionally, multi-metric theories were constructed as a generalisation of a ghost-free massive gravity. It is shown here explicitly that the existence of stable massive gravity is not necessary for consistent interacting multi-gravity theories.

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